Fatigue Reliability of Multidimensional Vibratory Degrading Systems under Random Loading

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Abstract: In this paper, the basic methodology for the fatigue reliability assessment of randomly vibrating multidegree-of-freedom systems is presented within the coupled response-degradation model. The fatigue process in the system components is quantified by the fatigue crack growth equations which—via the stress range—are coupled with the system response. Simultaneously, the system dynamics is affected by fatigue process via its stiffness degradation so that it provides the actual stress values to the fatigue growth equation. In addition to the general coupled response-degradation analysis, its special case of noncoupled fatigue crack growth is treated as well for the wide-band stationary applied stress by the use of its first four spectral moments and the approximate, empirically motivated, Dirlik's probability distribution for the stress range. Both, the general analysis and the illustrating exemplary problems elaborated in the paper provide the route to the fatigue reliability estimation in complex–hierarchical vibratory systems under random loading.

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Introduction

In the last years an increasing amount of research efforts has been directed toward stochastic modeling of various deterioration (or degradation) processes in mechanical/structural components. Because of the practical importance of fatigue damage and fracture in various engineering structures, stochastic models of fatigue accumulation have been a subject of special interest [Sobczyk and Spencer (1992) and references therein]. It should be underlined, however, that though the fatigue process is inherently associated with vibrations of mechanical/structural systems, the research in random vibration theory and in modeling of fatigue has been conducted without a proper mutual coupling. Stochastic analysis of dynamics of mechanical/structural systems has been focused on the characterization of the response (and its unsafe states, e.g., instability regions, first-passage probabilities), whereas the analysis of fatigue deterioration has been concentrated on the fatigue crack growth analysis assuming that the characteristics of the response (e.g., stresses) are given.

It is clear that a more adequate approach should account for

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the joint (coupled) treatment of both the system dynamics and deterioration process (e.g., fatigue accumulation). Such an analysis allows to account the effect of stiffness degradation during the vibration process on the response and, at the same time, gives the actual stress values for estimation of fatigue. It seems that in stochastic dynamics the coupled analysis of the response and degradation had been treated first in the context of elastoplastic (hysteretic) systems (Roberts 1978; Wen 1986). In the articles cited a degradation of the system is defined in terms of the hysteretic energy dissipation. As far as the joint analysis of random vibrations and fatigue degradation is concerned, one should mention the paper (Grigoriu 1990) containing the model in which fatigue crack growth equation is coupled with the equation for the amplitude of the response [obtained via the averaging method-Sobczyk (1991) and Soong and Grigoriu (1993)], and more extensive studies published in papers (Sobczyk and Trebicki 1999,2000; Gaidai et al. 2008). In all these papers the main attention is focused on single-degree-of-freedom (DOF) systems.

The objective of this work is an analysis of the responsedegradation problem of random vibration of structural/mechanical systems in a more general setting—for multi-DOF systems, multidimensional nature of the degradation (fatigue) process, and wide-band responses (in specific uncoupled problems). Such an analysis is inspired by the recently growing industrial interest in prediction of the response and fatigue degradation of large scale mechanical and structural systems; an important class of such systems includes a "hierarchy" of oscillatory subsystems with different fatigue-degrading stiffnesses.

We start from a general nonlinear formulation of both dynamics and degradation; however, to achieve a satisfactory level of clarity and effectiveness we focus our further analysis on linear dynamics (with a nonlinear fatigue degradation process). Also, we treat the system parameters, initial states of the system and degradation as given, deterministic quantities, although one is not always able to determine them exactly. In fact, the methodology and results in this paper are conditioned on the values of the

parameters that are involved into the model, initial conditions and the stochastic excitation process. We understand that the appropriate uncertainty and sensitivity analysis can, in general, be performed according to the existing methods—cf. among others: Igusa and Der Kiureghian (1988), Kotulski and Sobczyk (1987), and Schueller (2007).

Response-Degradation Models

General Governing Stochastic Differential Equations

Stochastic governing equations for many engineering dynamical systems should be represented in the form which accounts for both—the system dynamics and degradation process taking place in the system. In the case of mechanical/structural systems these are, above all, the elastic-plastic vibratory systems under severe random loadings in which the restoring force has a hereditary nature (Lin and Cai 1995; Wen 1986) and elastic systems with stiffness degradation due to fatigue damage.

In general, a coupled response-degradation model for nonlinear vibratory systems with random excitation (parametric and/or external) can be formulated in the following vectorial form:

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}\dot{\boldsymbol{y}}(t) + \boldsymbol{R}[\boldsymbol{y}(t), \dot{\boldsymbol{y}}(t), \boldsymbol{d}(t), \boldsymbol{X}_{1}(t, \gamma)] = \boldsymbol{P}\boldsymbol{X}_{2}(t, \gamma)$$
(1)

$$\mathcal{F}\{\mathbf{y}(t), \dot{\mathbf{y}}(t), \mathbf{d}(t), \mathbf{X}_1(t, \gamma)\} = \mathbf{0}$$
(2)

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_{1,0}, \quad d(t_0) = d_0$$
 (3)

where *M* and *C* represent the constant mass and damping matrices; respectively, $\mathbf{y}(t) = [y_1(t), y_2(t), \dots, y_N(t)] =$ unknown response vector process; *R* characterizes a nonlinear restoring force depending on \mathbf{y} and $\dot{\mathbf{y}}$, and on the process $d(t) = [d_1(t), d_2(t), \dots, d_M(t)]$, which characterizes a process responsible for degradation phenomena; $X_1(t, \gamma), X_2(t, \gamma)$ are given random processes symbolizing parametric and external excitations, respectively. The variable γ is an element of the space of elementary events in the basic scheme (Γ, F, P) of probability theory (Sobczyk 1991). $\mathcal{F}\{\cdot\}$ denotes a relationship between degradation and response processes; its specific mathematical form depends on the particular physical/mechanical situation. It is clear that $\mathbf{y}_0, \mathbf{y}_{1,0}, \mathbf{d}_0$ are given initial values of the response $[\mathbf{y}(t), \dot{\mathbf{y}}(t)]$ and degradation d(t) processes, respectively.

It should be noted, that in the cases when the original system is of a continuous type (e.g., beam, plate, shell) governed by partial differential equations, the Model (1)–(3) is a spatially discretized version (e.g., via Galerkin or finite element methods) of the original equations and it describes the system response-degradation as a function of time at fixed spatial points. It is also worth noticing that the meaning of $\mathcal{F}\{\cdot\}$ in Eq. (2) can be quite different in specific situations: it can be a differential operator, and also a functional defined on $[\mathbf{y}(t), \dot{\mathbf{y}}(t)]$. It is natural to assume that $d(t_0) = 0$. During the dynamical process vector d(t) approaches, as time increases, the unsafe state symbolized by the boundary \tilde{B} . Each $d \in B$ denotes a critical level of degradation. Set B of the admissible values of d(t)—being a part of the first quadrant constitutes a quality space. Therefore, the reliability of the system in question is defined as the probability that process d(t) will belong to B, i.e.





Fig. 1. MDOF system with cracks

Specific Vibratory Systems with Stiffness Degradation

An important class of vibration-degradation Model (1) has the form

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}\dot{\boldsymbol{y}}(t) + \boldsymbol{R}\{\boldsymbol{y}(t), \boldsymbol{k}[\boldsymbol{d}(t)]\} = \boldsymbol{P}\boldsymbol{X}(t,\gamma)$$
(5)

where $M = \text{diag}(m_p) \in \mathbb{R}^{N \times N}$, p = 1, 2, ..., N, $C \in \mathbb{R}^{N \times N}$, $k(d) = [k_1(d_1), \cdots, k_N(d_N)]^T \in \mathbb{R}^N$, with $k_p(d_p)$ be a function (empirically identified) characterizing dependence of *p*th stiffness element on the degradation mode d_p (e.g., it can be fatigue crack size, amount of wear, etc.); $\mathbf{R} \in \mathbb{R}^N$ is the nonlinear restoring force depending on \mathbf{y} and the degrading stiffness k(d); $\mathbf{P} \in \mathbb{R}^{N \times M}$ is a matrix that associates the external loads in $\mathbf{X}(t, \gamma) = [X_1(t, \gamma), X_2(t, \gamma), \cdots, X_N(t, \gamma)]^T \in \mathbb{R}^M$ to the DOF of the structure. In the linear case, the vector function $\mathbf{R}[\mathbf{y}, \mathbf{k}(d)]$ is a linear combination of the components $y_p(t)$ of $\mathbf{y}(t)$ with coefficients $k_p(d_p)$.

In particular, Model (5) includes the special class of multi-DOF hierarchical system, shown in Fig. 1. This class consists of a "perpendicular chain" of oscillatory systems axially subjected to random loading. The system in Fig. 1 consists of N bodies with the *p*th body having mass m_p . The p-1 and the *p* bodies are connected by elastic plate elements which provide the stiffness k_p to the system. It is assumed that in each plate element a fatigue crack develops perpendicular to the direction of the motion as shown in Fig. 1. The initial crack size of the plate element p is $L_{p,0}$. In general, it can be assumed that the axial stiffness provided by each plate depends on the crack size L_p . This dependence is introduced by letting the stiffness $k_p(L_p)$ be a function of the crack size L_p . The model in Fig. 1 can be used as a simplified model for a p-story shear building subjected to some lateral external excitation such as wind forces or base earthquake acceleration with various rates of damage growth at each level. Also, the two DOF version of Fig. 1 can be used to represent the dynamics of quarter car models with linear or nonlinear stiffness and damping characteristics (Papalukopoulos and Natsiavas 2007). Therefore, in what follows, such "hierarchical" systems will be of our main concern.

To represent Eqs. (5) in the state space form we define the 2N-dimensional state vector

$$z(t) = [y_1(t), y_2(t), \dots, y_N(t); \dot{y}_1(t), \dot{y}_2(t), \dots, \dot{y}_N(t)]^T = [y^T; \dot{y}^T]^T$$
(6)

Eqs. (5) can now be written in the form of a system of 2N equations of first order. This system can be written in the vectorial form as

$$\dot{z} = \begin{cases} [\boldsymbol{\theta}_{N,N}\boldsymbol{I}_{N,N}]\boldsymbol{z} \\ -\boldsymbol{M}^{-1}[\boldsymbol{\theta}_{N,N}\boldsymbol{C}]\boldsymbol{z} - \boldsymbol{M}^{-1}\boldsymbol{R}[\boldsymbol{z},\boldsymbol{k}(\boldsymbol{d})] \end{cases} + \begin{cases} \boldsymbol{\theta}_{N,M} \\ \boldsymbol{M}^{-1}\boldsymbol{P} \end{cases} \boldsymbol{X}$$
(7)

where $\boldsymbol{\theta}_{N,M} \in \mathbb{R}^{N \times M}$ = matrix of zeroes and $\boldsymbol{I}_{N,N} \in \mathbb{R}^{N \times N}$ = identity matrix.

Let us confine our analysis in this paper to the linear relationships between the system components. In this situation, System (5) is linear, with R[y, k(d)] = K[k(d)]y, where K[k(d)] is the stiffness matrix of the structure that depends on the individual degrading stiffnesses k(d), the Eq. (7) is linear, and the general vectorial equation of motion is

$$\dot{z}(t) = Az(t) + BX(t,\gamma)$$
(8)

where the matrix A is composed of the constant damping matrix C and the degrading stiffness vector k(d), as follows:

$$\mathbf{A} \equiv \mathbf{A}[\mathbf{C}, \mathbf{k}(\mathbf{d})] = \begin{cases} \mathbf{0}_{N,N} & \mathbf{I}_{N,N} \\ -\mathbf{M}^{-1}\mathbf{K}[\mathbf{k}(\mathbf{d})] & -\mathbf{M}^{-1}\mathbf{C} \end{cases}$$
(9)

and **B** is the matrix

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{\theta}_{N,M} \\ \boldsymbol{M}^{-1} \boldsymbol{P} \end{bmatrix}$$
(10)

As a specific case, assuming linear elastic behavior of the plate elements in Fig. 1, the stiffness matrix is

$$\boldsymbol{K}(\boldsymbol{d}) = \begin{bmatrix} k_1(d_1) + k_2(d_2) & -k_2(d_2) & \cdots & 0\\ -k_2(d_2) & k_2(d_2) + k_3(d_3) & \cdots & \vdots\\ \vdots & \vdots & \vdots & -k_N(d_N)\\ 0 & \cdots & -k_N(d_N) & k_N(d_N) \end{bmatrix}$$
(11)

The stiffness elements in k(d) are varying in time due to variability of d(t) in time. The analysis of the Systems (8)–(11) depends crucially on the mechanisms of degradation d(t). Therefore, the considered vibrating system governed by Eqs. (8)–(11) is a time-variant and, in general, the response z(t) is a nonstationary random process, even when $X(t,\gamma)$ is stationary. It should be noted, however, that the stiffness degradation is a process much slower than that of the system dynamics. In what follows we assume that stiffness degradation is due to fatigue taking place in the system elements and manifesting itself in fatigue crack growth during the vibration process. Functions $k_p(d_p)$ are assumed to be nonincreasing functions known from the empirical data (Sobczyk and Trebicki 1999).

We wish to note that the simplification made in this section (linear system) will allow to perform further analysis consistently via equations for the covariance matrix Q(t) of the state vector [cf. Eq. (27)]. The idea of using differential equations for statis-

tical moments can also be extended to nonlinear System (5), but in this case one obtains the hierarchy of coupled equations for moments of different orders which makes some methodical troubles [cf. Sobczyk (1991)]; although various approximations of this hierarchy are possible, the computational work is usually involved.

Fatigue-Induced Degradation

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In the analysis of response of vibrating systems with stiffness degradation due to fatigue it is natural to quantify the process d(t) in Eq. (1) by scalar processes $d_p(t)$ which are deliverable from the "kinetic" crack growth equations. These equations contain the stress intensity factor range $\Delta K = K_{\text{max}} - K_{\text{min}}$. A wide class of the fatigue crack growth models can be represented by the Paris equation (Sobczyk and Spencer 1992) governing the evolution of the crack length $L_p(t)$ at the plate p as

$$\frac{dL_p}{dt} = C'_p (\Delta K)^{\mu_p} = C'_p [B_p(L_p)]^{\mu_p} [\Delta S_p]^{\mu_p}$$
(12)

where μ_p , C'_p =empirical constants (and C'_p includes the equivalent frequency); $B_p(L_p)$ is the factor which accounts for the crack length and shape of the specimen and crack geometry, and ΔS_p —the stress range, may depend on time. In the problem considered here the "specimens with cracks" are the finite rectangular plates, so $B_p(L_p)$ can be taken in the form (Miannay 1998)

$$B_{p}(L_{p}) = \frac{1}{b_{p}} \sqrt{\pi L_{p}} \left(\sqrt{\cos \frac{\pi L_{p}}{b_{p}}} \right)^{-1}, \quad \frac{L_{p}}{b_{p}} < 0.7$$
(13)

where b_p =width of the *p*th plate element. The second factor in Eq. (12) is the μ_p -power of the stress range (generated in the vibrating element), i.e.

$$\Delta S_p = S_{p,\text{max}} - S_{p,\text{min}} \tag{14}$$

which has to be evaluated as a result of solving multidimensional vibration problem for $\mathbf{y}(t) = [y_1(t), \dots, y_N(t)]^T$, since equations for $\mathbf{y}_p(t)$, $p = 1, 2, \dots, N$ are coupled. In general, ΔS_p , $p = 1, 2, \dots, N$ are correlated, but in our analysis this correlation is assumed to be negligible. We also assume that the mean stress is zero.

Characterization of Random Stress Range Using Spectral Moments

Characterization of the random stress range (14) constitutes a crucial part of the analysis. In the existing works, most often ΔS_p was characterized by the envelope of the stress generated by the scalar response process and (for linear systems) the Rayleigh probability distribution (PD). However, the concept of the envelope itself cannot be defined for all random processes. Only for stationary narrow-band processes it has clear meaning, and when process in question is Gaussian, the probability density function (PDF) of the envelope has the Rayleigh distribution. These are serious restrictions if one has in mind a wider class of practical applications, e.g., stationary wide-band processes. The characteristic, which is quite reasonable in practice, is the mean range $S_{\rm mr}$

$$S_{\rm mr} = E[\Delta S] = E[S_{\rm max}] - E[S_{\rm min}]$$
(15)

where $S_{\text{max}}=m_S+P$, $S_{\text{min}}=m_S-P$, $m_S=E[S]$ and P=random height of peaks. Therefore the mean range is $S_{\text{mr}}=2E[P]$. For the stationary and Gaussian processes it is as follows (Sobczyk and Spencer 1992):

$$S_{\rm mr} = 2\sigma_S \sqrt{\frac{\pi}{2}(1-\varepsilon^2)} \tag{16}$$

where σ_S =root mean square of S(t) and ε =spectral width parameter

$$\varepsilon = (1 - \alpha^2)^{1/2}, \quad \alpha = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}}$$
 (17)

and λ_0 , λ_2 , and λ_4 =spectral moments (SMs) of S(t) and $\sigma_S = \sqrt{\lambda_0}$. For wide-band processes $\varepsilon > 0$; if the process is a narrow band one, then $\varepsilon \to 0$ and $S_{\rm mr} = \sqrt{2\pi\sigma_S}$. The SMs λ_k of the stress process S(t) are defined as the integral

$$\lambda_k = \int_{-\infty}^{\infty} |\omega|^k g_s(\omega) d\omega \tag{18}$$

over infinite range, where $g_S(\omega)$ =spectral density of S(t). Thus, the SMs may not be finite. The moment λ_{2k} is finite if and only if the correlation function $K_s(\tau)$, $\tau = t_2 - t_1$, possesses a derivative of order 2k at $\tau = 0$ (Cramer and Leadbetter 1967). Note that the SMs λ_0 , λ_2 , and λ_4 of the stress process S(t) are respectively the second moments of S(t), $\dot{S}(t)$, and $\ddot{S}(t)$.

Characterization of Random Stress Range Using Probability Distribution

The averaged characteristics of the stress field (such as S_{mr}) themselves do not reflect specific features of the stress spectrum (e.g., bimodal spectral densities) nor the properties of the PD of ΔS_p for wide-band processes. However, as it has recently been shown by Dirlik (1985) [see also Bishop and Sherrat (1990) and Benasciutti and Tovo (2005)], the SMs $\lambda_0, \lambda_1, \lambda_2, \lambda_4$ can constitute a base for construction of the approximate closed-form formula for the probability density of the stress range (in the rain-flow cycle analysis). This semiempirical probability density being a mixture of one exponential and two Rayleigh distributions has been derived by fitting the shape of a rain-flow range distribution via minimizing the normalized error between the rain-flow ranges and the density model (where range is meant as a stress difference between a peak and the next valley of the stress trajectory). Its form is as follows

$$p(\Delta S) = \frac{\frac{D_1}{Q}e^{-Z(\Delta S)/Q} + \frac{D_2 Z(\Delta S)}{R^2}e^{-Z^2(\Delta S)/2R^2} + D_3 Z(\Delta S)e^{-Z^2(\Delta S)/2}}{2\sqrt{\lambda_0}}$$

where

$$Z(\Delta S) = \frac{\Delta S}{2\sqrt{\lambda_0}} \tag{20}$$

and D_1 , D_2 , D_3 , R, and Q=specific algebraic functions of the SMs λ_0 , λ_1 , λ_2 , λ_4 , given by

$$D_1 = \frac{2(x_m - \alpha^2)}{1 + \alpha^2}, \quad D_2 = \frac{1 - \alpha - D_1 + D_1^2}{1 - R}, \quad D_3 = 1 - D_1 - D_2$$
(21)

$$Q = \frac{1.25(\alpha - D_3 - D_2 R)}{D_1}, \quad R = \frac{\alpha - x_m - D_1^2}{1 - \alpha - D_1 + D_1^2}$$
(22)

$$x_m = \frac{\lambda_1}{\lambda_0} \left[\frac{\lambda_2}{\lambda_4} \right]^{1/2} = \alpha \beta, \quad \alpha = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}}, \quad \beta = \frac{\lambda_1}{\sqrt{\lambda_0 \lambda_2}}$$
(23)

This formula can be interpreted as "empirical" or simulation inspired extension of the Rayleigh distribution to non-narrowband processes. It can be viewed as an effective tool for fatigue crack estimation under wide-band stationary applied stress. It will be shown in the "Noncoupled Response-Degradation Problem: Via Dirlik's Approximation of Stress Range" section that formula (19) can be used for fatigue predictions in the case of noncoupled response-degradation problem, where the stress range in fatigue crack growth equation is taken from the stationary (and wideband) response of multidimensional system. It is worth adding that in the last years some attempts have been made to extend the Dirlik's formula (19) to non-Gaussian processes (stresses) via replacement of its third term (of Rayleigh-type) by log-normal function (Wang and Sun 2005).

Crack Growth Prediction for Hierarchical Systems

The equations for the evolution of the crack length are given by the Paris law (12). Thus, the evolution of the crack length depends on the description of the stress range ΔS_p . In the uncoupled case—when the stiffnesses k_p in the vibratory components are regarded to be constant and the load is a stationary random process—the stress ranges ΔS_p will be described in terms of SMs (16) and (17) of the stationary response or by making use of the Dirlik's formula (19) for the probability density of ΔS_p . The required SMs are calculated from the solution of the governing vibratory equations.

Although characterization of fatigue loads/applied stress is usually based on stationary random processes, in the coupled response-degradation problem the response of vibratory system, due to variability of the stiffness, is generally nonstationary. So, instead of SMs, the stress range will be characterized by the timevarying root mean square, i.e., in Eq. (12) $\Delta S_p(t) = \sqrt{2\pi\sigma_p}(t)$. In the coupled problem the standard deviations $\sigma_p(t)$ occurring in fatigue crack growth Eq. (12) are coupled with the moment equations for the system response, e.g., with the equations for the covariance $Q_z(t)=E[z(t)z^T(t)]$ of the state vector z governed by the system of first order Eq. (7).

In what follows, the first subsection presents the analysis for the prediction of fatigue for the coupled response-degradation case. The second and third subsections present the analysis for the prediction of fatigue for the noncoupled response-degradation case via SMs and Dirlik's approximation of the stress range, respectively.

Coupled Response-Degradation Problem

When stiffness degradation takes place during the vibration process, the linear system (8) has a time-varying matrix A = A(t) since

$$\boldsymbol{k}(\boldsymbol{d}) \equiv \boldsymbol{k}[\boldsymbol{L}(t)] \tag{24}$$

and thus, using Eq. (9)

$$\mathbf{A}(t) \equiv \mathbf{A}[\mathbf{L}(t)] \tag{25}$$

Assuming that the stress range $\Delta S_p(t)$ is characterized by the time-varying root mean square as $\Delta S_p(t) = \sqrt{2\pi\sigma_p(t)}$, the evolution of the crack length L(t) obtained by solving Eq. (12) is a deter-

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(19)

ministic function of time. Thus, A(t) is also a deterministic function of time *t*.

Let us consider the situation when the vectorial load process $X(t,\gamma)$ is a white noise with intensity $G_0(t)$, i.e., $X(t,\gamma) = W(t,\gamma)$, where $W(t,\gamma)$ has zero mean and covariance

$$\boldsymbol{R}_{W}(t,s) = E[\boldsymbol{W}(t)\boldsymbol{W}^{T}(s)] = \boldsymbol{G}_{0}(t)\delta(t-s)$$
(26)

In this case, the covariance matrix $Q_z(t) = E[z(t)z^T(t)]$ of the state vector z(t) for s=t is given by the following system of equations (called sometimes the Lyapunov equations) (Soong and Grigoriu 1993; Lutes and Sarkani 2003)

$$\frac{d\boldsymbol{Q}_{z}(t)}{dt} = \boldsymbol{A}[\boldsymbol{L}(t)]\boldsymbol{Q}_{z}(t) + \boldsymbol{Q}_{z}(t)\boldsymbol{A}[\boldsymbol{L}(t)]^{T} + \boldsymbol{B}\boldsymbol{G}_{0}(t)\boldsymbol{B}^{T}$$
$$\boldsymbol{Q}_{z}(t_{0}) = \boldsymbol{Q}_{0}$$
(27)

where Q_0 =covariance of stationary response matrix of the initial nondegraded state obtained by solving the system (27) for constant stiffness matrix $K_0 = K[k(L_0)]$. The system (27) is coupled with the system of degradation Eq. (12) for the *p*th components of the crack length, $L_p(t)$, p=1,2,...,N. The initial conditions for the Eq. (12) are: $L_p(t_0) = L_{p,0}$, p=1,2,...,N.

The formulation can be readily extended for the more general situation of filtered white-noise excitation that can be modeled as the solution of a system of linear differential equations to white-noise input. In this case, the state vector z(t) can be augmented to include the states of the linear differential equations describing the nonwhite-noise input. Thus, a similar Lyapunov set of equations of the form (27) holds for the combined system with states describing the structural response states and the filter states associated with the input.

The information required in Eq. (12) is the axial stress range perpendicular to the crack in *p*th plate element, which for the linear hierarchical structure in Fig. 1, can be written in the form $S_p(t) = k_p(L_p)[y_p(t) - y_{p-1}(t)]$. In compact form, the axial stress $S_p(t)$ can be written in terms of the response vector $\mathbf{y}(t)$ as $S_p(t)$ $= (\mathbf{\delta}_p^T - \mathbf{\delta}_{p-1}^T)\mathbf{y}$, where $\mathbf{\delta}_p$ is a vector that has the *p* element equal to one and all other elements equal to zero. Letting $\mathbf{S}(t)$ $= [S_1(t) \cdots S_N(t)]^T$ be the vector of axial stresses in the elastic plate elements, one can relate the axial stress vector to the response vector $\mathbf{y}(t)$ from the compact relationship

$$\mathbf{S}(t) = \mathbf{F}(L)\mathbf{y}(t) \tag{28}$$

where

$$F(L) = \operatorname{diag}[k(L)](I_{N,N} - I_{N,N}^*)$$
(29)

is a matrix that for stiffness degradation problems depends on the vector of the crack lengths L(t); $I_{N,N} = [\delta_1, \dots, \delta_N]^T = \text{identity matrix}$ of dimension N; and $I_{N,N}^* = [\delta_0, \dots, \delta_{N-1}]^T \in \mathbb{R}^{N \times N} = \text{matrix}$ having the entries immediately below the diagonal equal one and all other entries equal to zero. Since $\Delta S_p(t)$ in Eq. (12) is characterized by the root mean square of the random stress, i.e., $\Delta S_p(t) = \sqrt{2\pi\sigma_p}(t)$, we need to obtain a direct relationship between $\sigma_p(t)$ and the components of the covariance matrix $Q_z(t)$. This is achieved by using Eq. (28) and noting that $\sigma_p^2(t), p=1, \dots, N$, are the diagonal elements of the matrix

$$E[S(t)S^{T}(t)] = F(L)Q_{v}(t)F^{T}(L)$$
(30)

where $Q_{y}(t) = N \times N$ upper left partition of the matrix $Q_{z}(t)$.

The system of N crack growth Eq. (12) and the system (27) of the response covariance form a system of coupled nonlinear differential equations that have to be solved simultaneously since the

crack length increase affects the stiffness of each element and therefore the covariance of the response. The set of Eq. (30) are auxiliary equations needed to compute the mean square of $\sigma_p^2(t)$ used in the characterization of the random stress range $\Delta S_p(t) = \sqrt{2\pi\sigma_p(t)}$ involved in Eq. (12), in terms of the covariance response matrix $Q_z(t)$ derived from Eq. (27).

The system of coupled differential equations is stiff due to the slow evolution process associated with the crack growth and the fast evolution process associated with the dynamics of the structure. Thus, the solution of the system of coupled differential equations is obtained using the Gear numerical differentiation formula (Gear 1971) suitable for solving stiff differential equation problems.

Noncoupled Response-Degradation Problem: Via Spectral Moments

Next, it is assumed that crack growth does not significantly affect the axial stiffness of the plate elements so that the stiffnesses remain constant, independent of the crack size, that is $k_p(L_p) = k_{p,0} = \text{const}$ or, equivalently, $\mathbf{k}(L) = \mathbf{k}_0$ is a constant vector independent of the evolution of the vector $\mathbf{L}(t)$ of crack lengths. In this case the state space matrix $\mathbf{A}(t) \equiv \mathbf{A}$ is constant, as well as the matrix $\mathbf{F}(L) \equiv \mathbf{F}$ is constant, independent of the crack sizes $L_p(t)$, $p=1, \dots, N$. Simply, it is assumed that the stress range ΔS_p in each plate element [in degradation Eq. (12)] is specified by Eqs. (16) and (17), i.e., $\Delta S_p = S_{p,\text{mr}}$. That is, the stress range is completely specified by the SMs of the stress process within each plate.

In this case, the Eqs. (27) for the covariance response of the state vector of the system are uncoupled from the crack growth or degradation Eqs. (12). Specifically, the solution for the crack growth proceeds as follows. The linear equations of motion in the state space form are used to obtain the covariance matrix $Q_z = E[z(t)z^T(t)]$ of the state vector by solving the corresponding Lyapunov system of equations which for stationary state has the form

$$AQ_z + Q_z A^T + BG_0(t)B^T = 0$$
(31)

with constant matrix $A[L(t)] \equiv A$ corresponding to the constant nondegrading stiffness properties $k_{p,0}$, $p=1, \dots, N$, of the elastic plate elements. The solution can be carried out numerically. Noting that S(t)=Fy(t), $\dot{S}(t)=F(L)\dot{y}(t)$ and

$$\ddot{\mathbf{S}}(t) = \mathbf{F} \ddot{\mathbf{y}}(t) = \mathbf{F} [-\mathbf{M}^{-1} \mathbf{K} \mathbf{y}(t) - \mathbf{M}^{-1} \mathbf{C} \dot{\mathbf{y}}(t) + \mathbf{M}^{-1} \mathbf{P} \mathbf{X}(t)] \quad (32)$$

the elements of the covariance matrix are used to find the covariance Q_{η} of the vector $\eta(t) = [S^T(t)\dot{S}^T(t)]^T$ of the stress responses and their derivatives within each plate from the relationship

$$\boldsymbol{Q}_{\boldsymbol{\eta}} = \boldsymbol{H} \boldsymbol{Q}_{\boldsymbol{z}} \boldsymbol{H}^{T} \tag{33}$$

where H is given by

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{\theta}_{N,N} & \boldsymbol{\theta}_{N,M} \\ \boldsymbol{\theta}_{N,N} & \boldsymbol{F} & \boldsymbol{\theta}_{N,M} \\ -\boldsymbol{F}\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{F}\boldsymbol{M}^{-1}\boldsymbol{C} & \boldsymbol{F}\boldsymbol{M}^{-1}\boldsymbol{P} \end{bmatrix}$$
(34)

The SMs $\lambda_{i,p}$, i=0,2,4 of the stress process involved in Q_{η} are then obtained and used in Eq. (12) with $\Delta S_p = S_{p,\text{mr}}$ given by Eqs. (16) and (17). The solution for the crack growth length $L_p(t)$ as a function of time is computed by numerically solving the first order differential Eq. (12). Alternatively, the SMs $\lambda_{i,p}$, i=0,2,4

can be computed from the one dimensional integrals (18). This requires numerical integration to be carried out over an infinite domain of ω and is usually more tedious computationally.

It should be emphasized that the formulation in Eqs. (33) and (34) is applicable for the case for which the excitation X(t) is a filtered white-noise excitation given by a system of ordinary differential equations in which case y, \dot{y} , and the filter states X are part of the state vector z. For the white-noise excitation X(t) = W(t) the SM $\lambda_{4,p}$ takes infinite values. The formulation still holds if the contribution of the spectral width parameter $\varepsilon(t)$ is ignored in Eqs. (12) and (16) by setting $\varepsilon(t)=0$. Finite values of the SMs $\lambda_{4,p}$ can be obtained by using a process resembling white noise with constant spectral density in the interval $[-\omega_0, \omega_0]$ and zero spectral values outside this interval. This truncated white-noise process is often used to carry out the integration in (18) with bounded limits $[-\omega_0, \omega_0]$ without affecting the values of the spectral densities $\lambda_{0,p}$ and $\lambda_{2,p}$, provided that ω_0 is high enough.

Noncoupled Response-Degradation Problem: Via Dirlik's Approximation of Stress Range

Let us consider now the fatigue crack growth prediction making use of Dirlik's formula (19) for the PDF of ΔS_p . For convenience, it is assumed that crack growth does not significantly affect plate element stiffness so that the stiffnesses remain constant and equal to $k_p(L_p) = k_{p,0}$. The PDFs for ΔS_p are completely defined from the SMs $\lambda_{0,p}$, $\lambda_{1,p}$, $\lambda_{2,p}$, and $\lambda_{4,p}$ of the axial stress response and its derivatives. These SMs can be computed by the integral in Eq. (18) which can be used with bounded limits $[-\omega_0, \omega_0]$ to compute λ_4 in the case of white-noise input. Alternatively, the SMs $\lambda_{i,p}(t) = \lambda_{i,p}$, i=0,2,4 involved in Q_{η} can be directly computed by solving the Lyapunov Eq. (31) for the covariance response Q_z of the state vector and then using the relationship (33).

Given the PDFs for ΔS_p from the Dirlik formula, the predictions of the PDFs of the crack size $L_p(t) \equiv L_p(t; \Delta S_p)$ are obtained from the Eqs. (12). These PDFs can then be used to obtain the characteristics of failure, such as the mean and the variance of failure time, the probability of failure at a given time, etc. For demonstration purposes, failure $F_p(t)$ in the plate element p is defined as the state in which the crack length $L_p(t; \Delta S_p)$ exceeds a critical value $L_{p,crit}$ in a given time interval [0, t], that is

$$F_p(t) = \{L_p(t; \Delta S_p) > L_{p, \text{crit}}\}$$
(35)

The probability of failure $Pr[F_p(t)]$ in the plate element p is given by the integral

$$\Pr[F_{p}(t)] = \int_{L_{p}(t;\Delta S_{p}) \ge L_{p,crit}} p(\Delta S_{p}) d(\Delta S_{p})$$
$$= \int_{\Delta S_{p} \ge \Delta S_{p,crit}(t)} p(\Delta S_{p}) d(\Delta S_{p}) = \int_{\Delta S_{p,crit}(t)}^{\infty} p(\Delta S_{p}) d(\Delta S_{p})$$
(36)

where $p(\Delta S_p)$ =PDF given by Eq. (19) and $\Delta S_{p,crit}(t)$ is the value of the stress range ("design point" in reliability terminology) that can be calculated for given time instant *t* by solving the equation

$$L_p(t;\Delta S_p) = L_{p,\text{crit}} \tag{37}$$

with respect to ΔS_p . A numerical scheme can be used to obtain the solution of Eq. (37) for each time *t* with $L_p(t;\Delta S_p)$ given by the solution of Eq. (12). The integration in Eq. (36) is one dimensional and can be carried out efficiently using available numerical algorithms.

Numerical Illustrations

The methods proposed for the fatigue life predictions are applicable for the *N* DOF system shown in Fig. 1. For demonstration purposes, the system is subjected to a base acceleration $\ddot{a}(t)$. The base excitation is assumed to be stationary white noise, i.e., $\ddot{a}(t) = \mathbf{W}(t)$, with power spectral density equal to 10^{-2} . In this case, the matrix \mathbf{P} relating the excitation forces to the degrees of freedom of the systems takes the form $\mathbf{P}=-\mathbf{M}\mathbf{1}$, while the input excitation vector $\mathbf{X}(t)$ takes the form $\mathbf{X}(t)=\ddot{a}(t)=\mathbf{W}(t)$, where **1** is defined to be a vector with all elements equal to one. For this mathematically defined white noise, the spectral parameter λ_4 is infinite.

From the computational point of view, the random excitation is considered to have a constant power spectral density over the frequency range $[-\omega_0, \omega_0]$ which contains the values of the frequencies of the main contributing modes of the system. The SMs are then computed using Eq. (18) with the domain of the integration to be $[-\omega_0, \omega_0]$ for sufficient high value of ω_0 . The results from the integration for λ_0 and λ_2 are same as the ones obtained by solving the Lyapunov equation for the covariance response. The results of the integration for computing λ_4 depend on the value of ω_0 indicating the range of spectral frequencies with significant energy.

In the numerical results presented, the methodologies used are termed "constant stiffness—SM" method referring to the noncoupled response-degradation problem in subsection "Noncoupled Response-Degradation Problem: Via Spectral Moments," "constant stiffness—PD" method referring to the noncoupled response-degradation problem based on Dirlik's formula for the PD of the stress range in subsection "Noncoupled Response-Degradation Problem: Via Dirlik's Approximation of Stress Range," and "stiffness degradation" method referred to the coupled response-degradation problem in subsection "Coupled Response-Degradation Problem."

Single Degree of Freedom System

The case of a single oscillator (N=1) is first considered. The initial crack length is assumed to be equal to $L_0=10^{-2}$. Also the values of C' and μ , defining the degradation equations, are assumed to be $C'=1.03\times10^{-12}$ and $\mu=3.89$. The mass and the plate properties without the crack are selected so that the natural frequency of the system is 10 Hz. The damping coefficient is selected so that the damping ratio of the system is 5%. The value of ω_0 , defining the domain of integration of the SMs in Eq. (18), is taken to be $\omega_0=30$ Hz.

Constant Stiffness—Spectral Moments

Results for the crack length growth are first obtained for the constant stiffness—SM method. The evolution of the crack growth is obtained from Eq. (12), considering that the response has reached stationary state due to stationary white-noise excitation. The results for the crack length growth predictions in the system are shown in Fig. 2 for the cases of spectral width parameter $\varepsilon = 0$ and $\varepsilon \neq 0$. It can be seen that the inclusion of spectral width parameter ε in the model significantly affects the predictions of failure, prolonging the time to failure.

Constant Stiffness—Probability Distribution

Next, results for the constant stiffness—PD method are presented using Dirlik's formula (19) for the PDF of the stress range ΔS . This PDF for the N=1 system is shown in Fig. 3. Using this PDF, the probability of failure of the system is calculated for a certain



Fig. 2. Crack size growth with respect to time for $L_{0,p}=10^{-2}$ using the "constant stiffness—SM" method for the system N=1

critical value of $L_{1,\text{crit}} = 10^{-1}$ as shown in Fig. 4 for different values of the initial crack size L_0 . The results are also compared to the lifetime predictions provided by the constant stiffness—SM method for $\varepsilon = 0$ and $\varepsilon \neq 0$.

For demonstration purposes, consider the case in Fig. 4 for which the initial crack size equals to $L_0 = 10^{-2}$. It can be seen that the failure time $t_{\text{fail}} = 1.26 \times 10^{10}$ s predicted from the constant stiffness—SM method with $\varepsilon \neq 0$ corresponds to very high failure probability Pr(F)=0.937 predicted by the constant stiffness—PD method. Moreover, the constant stiffness—PD method predicts that the time of failure that corresponds to failure probabilities Pr(F)=0.01, 0.1 and 0.5 equals to $t_{\text{fail}}=9.7 \times 10^5, 3.9 \times 10^6$ and 4.3×10^7 s, respectively. Similar interpretations can be inferred comparing the other cases shown in Fig. 4. In general, from the results in Fig. 4, one can conclude that more conservative estimates of failure times corresponding to small failure probabilities are obtained for the constant stiffness—SM method which correspond to failure probabilities very close to one.



Fig. 3. PDF of the stress range ΔS



Fig. 4. Probability of failure versus time for different initial crack sizes, along with comparisons of lifetime predictions from the "constant stiffness—SM" method for $\varepsilon = 0$ and $\varepsilon \neq 0$

Stiffness Degradation

Finally, the stiffness degradation method is considered for which the crack length affects the stiffness of the structure, i.e., the case which $k_p(L_p)$ depends on *L*. This effect can be introduced by employing the following empirical stiffness degradation function available in the literature (Sobczyk and Trebicki 1999)

$$k_p(L_p) = k_{0,p} \{\beta_1 + \beta_2 \exp[-\beta_3 (L_p/b_p)^{\beta_4}]\}$$
(38)

where b_p =width of the *p* plate. The values of the coefficients are selected to be $\beta_1=0.5$, $\beta_2=0.5$, $\beta_3=1$, and $\beta_4=1$ such that $k_p(0) = k_{p,0}$, where $k_{p,0}$ is the initial stiffness of the uncracked plate.

Numerical results are presented assuming that the initial crack size equals to $L_{0,p}=10^{-2}$. The crack growth predictions in this case are shown in Fig. 5 for the case of $\varepsilon = 0$ and are compared to the corresponding crack growth predictions obtained from the



Fig. 5. Comparison of crack growth prediction obtained from the "stiffness degradation" and the "constant stiffness—SM" methods $(\varepsilon = 0)$



Fig. 6. Variance ratio $\sigma_S^2/\sigma_{S,0}^2$ of the stress response with respect to time

constant stiffness—SM. As expected, lifetime reduces when the effect of stiffness degradation due to crack growth is taken into account in the formulation.

The ratio $\sigma_5^2/\sigma_{5,0}^2$ of the variance $\sigma_5^2(t)$ of the axial stress response S(t) obtained from the stiffness degradation method to the constant variance $\sigma_{5,0}^2$ obtained from the constant stiffness—SM method (nondegrading structure) is given in Fig. 6 as a function of time. Also, the stiffness reduction k[L(t)]/b with respect to time, due to degradation L(t), is shown in Fig. 7. It can be seen that the variance ratio increases, indicating that the response of the structure increases due to degradation. This increase has a result of accelerating failure which, as shown in Fig. 5, occurs earlier than the time expected for nondegrading constant stiffness structures.

Three-Degree-of-Freedom System

The methodology is next applied to a three-DOF hierarchical system (N=3), shown in Fig. 1. The initial crack length is assumed to be equal to $L_{0,p}=10^{-2}$ for the three subsystems. Also the values



10⁰ p=2, _E=0 p=3, _E=0 o=1, ε≠0 p=2, _{ɛ≠}0 Crack Growth L_b(t)/b_b p=3. ε≠0 10 10 10¹² 10⁰ 10² 10 10 10 10 Time (sec)

Fig. 8. Crack size growth with respect to time for the three-DOF system (N=3) using the "constant stiffness—SM" method

of C'_p and μ_p are assumed to be $C'_p = 1.03 \times 10^{-12}$ and $\mu_p = 3.89$, p=1,2,3. For the mass and plate properties selected, the natural frequencies of the three DOF system without cracks are 4.45, 12.47, and 18.02 Hz. The damping matrix C is chosen assuming that the system is classically damped at its initial nondegrading state. Specifically, the damping matrix C is selected so that the values of the modal damping ratios corresponding to the uncracked structure are 5% for all three modes. The value of the upper frequency ω_0 needed in computing λ_4 using Eq. (18) is taken to be $\omega_0=30$ Hz.

Constant Stiffness—Spectral Moments

Results for the crack growth at each plate element as a function of time for the constant stiffness—SM method are shown in Fig. 8 for the three subsystems and for the cases of $\varepsilon = 0$ and $\varepsilon \neq 0$. It can be seen that the crack grows faster on the first plate since the stresses in this plate takes higher values than the stresses in the other two plates. Also, the inclusion of the spectral width parameter ε ($\varepsilon \neq 0$) in the formulation significantly affects predictions of failure, prolonging the time to failure for the first and third subsystem.

Constant Stiffness—Probability Distribution

Finally, results for the constant stiffness—PD method are presented using Dirlik's formula (19) for the PDFs of the stress ranges ΔS . The PDFs for all axial stress ranges $\Delta S_p(t)$ are shown in Fig. 9. Using these PDFs, the probabilities of failure for the first, second and third subsystems are calculated for a certain critical value of $L_{p,crit}=10^{-1}$, p=1,2,3, as shown in Fig. 10 for initial crack size values $L_{p,0}=10^{-2}$. The results are also compared to the deterministic lifetime predictions provided by the constant stiffness—SM method for $\varepsilon=0$ and $\varepsilon \neq 0$.

For the predictions provided by the constant stiffness—PD method, it can be seen that for probability of failure of the system is controlled by the failure of the first subsystem since the time of failure for any probability level is smaller than the time of failure for the other two subsystems. Also, it can be seen that the failure time $t_{\text{fail}}=5.7 \times 10^9$ s predicted from the constant stiffness—SM method with $\varepsilon \neq 0$ corresponds to very high failure probability $\Pr(F)=0.82$ predicted by the constant stiffness—PD method.

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Moreover, the constant stiffness—PD method predicts that the time of failure that corresponds to failure probabilities Pr(F) = 0.01, 0.1 and 0.5 equals to $t_{fail} = 1.0 \times 10^5$, 5.1×10^5 , and 3.0×10^7 s, respectively. Similar interpretations can be inferred comparing the other cases shown in Fig. 10. In general, from the results in Fig. 10, one can conclude that more conservative estimates of failure times corresponding to small failure probabilities are obtained for the constant stiffness—PD method than the estimates provided by the constant stiffness—SM method which correspond to failure probabilities very close to one.

Stiffness Degradation

Next, results are presented for the stiffness degradation method for which the crack length affects the stiffness of the structure. This effect is introduced by employing the empirical stiffness degradation function (38) for each of the three plate elements. Numerical results are presented using that the initial crack sizes are all equal to $L_{0,p}=10^{-2}$, p=1,2,3. The crack growth predictions in this case are shown in Fig. 11 for the case of $\varepsilon=0$ and are



Fig. 10. Probability of failure versus the time of failure for the three subsystems, along with comparison of lifetime prediction from the "constant stiffness—SM" method for $\varepsilon = 0$ and $\varepsilon \neq 0$



Fig. 11. Comparison of crack growth prediction obtained from the "stiffness degradation" and the "constant stiffness—SM" methods $(\varepsilon = 0)$

compared to the crack growth predictions obtained from the constant stiffness—SM method. As expected, it can be seen that the lifetime reduces when the effect of stiffness degradation due to crack growth is taken into account in the formulation.

The ratio $\sigma_{S,p}^2/\sigma_{S,p0}^2$, p=1,2,3, of the variance $\sigma_{S,p}^2(t)$ of the axial stress response $S_p(t)$ obtained from the stiffness degradation method to the constant variance $\sigma_{S,p0}^2$ obtained from the constant stiffness—SM method (nondegrading structure) are shown in Fig. 12 as a function of time. Also, the stiffness reduction $k[L_p(t)]/b_p$ with respect to time, due to degradation $L_p(t)$, is shown in Fig. 13. It can be seen that the variance ratios increases for all axial stresses, indicating that degradation affects the response of the structure. The most pronounced increase is manifested in the first subsystem. This increase has a result of accelerating failure which occurs earlier for the first subsystem as compared to the time of failure expected for nondegrading structure.



Fig. 12. Variance ratio $\sigma_{S,p}^2/\sigma_{S,p0}^2$ of the stress response with respect to time



Fig. 13. Stiffness decrease $k_p(L_p(t))/k_{0,p}$ with respect to time

Conclusions

In this paper a general formulation and the effective method for predicting the fatigue lifetime in randomly vibrating linear multi-DOF systems/structures have been presented. The analysis is based on the coupled response-degradation model and it takes into account a wide-band spectrum of the stress process.

The fatigue process is characterized by crack growth in the structural components and is represented by Paris equation in which the stress range is evaluated form the multidimensional random response of the system. Both the stiffness degradation due to fatigue during the vibration, and nondegrading case are considered. The stress range was approximated by either the SMs or the empirically motivated and widely used Dirlik's PD. The prediction capabilities of the proposed analyses were demonstrated using special classes of single and multi-DOF structural systems. For the formulation based on SMs in the nondegrading case, it was demonstrated that the inclusion of the spectral width parameter in the model prolongs the time of failure of the system. For the formulation based on Dirlik's formula in the nondegrading case, more conservative estimates of failure times corresponding to small failure probabilities were obtained than the estimates provided by the SMs which correspond to failure probabilities very close to one. Finally, it was demonstrated that stiffness degradation accelerates failure due to fatigue in the various structural components.

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